Crossian Resonance

A Multidisciplinary Research Journal **ISSN 0976-5417**

(A refereed Biannual Published in June and December)

Vol. 15, No.2, December 2024

https://research.holycrossngl.edu.in/

Domination Number in Edge Product Cluster Hypergraphs

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ABSTRACT

The study on Domination in Edge Product Cluster Hypergraph aims for a balance between the concepts of Domination and Edge Product in Cluster Hypergraphs. The cluster hypergraph $H = (V_X, E)$ *is said to be an Edge Product Cluster Hypergraph if there exists an edge function* $f: E \to I$ *such that the edge function* f *and the corresponding edge product function* F *of* f *on* V_X *have the following conditions are* $F(v) \in I$ *for every* $v \in V_X$ *and if* $f(e_1) \times f(e_2) \times \ldots \times f(e_i) \in I$ *for some edges* $e_1, e_2, \ldots e_i \in E(H)$ *then the edges* $e_1, e_2, \ldots e_i$ *are all adjacent to some vertex set* V_X*. This article explores the concept of Domination in Edge Product Cluster Hypergraphs. Also, some theorems related to the concept of Domination in Edge product Cluster Hypergraphs have been discussed and demonstrated in this article.*

Keywords: *cluster hypergraph, edge product cluster hypergraph, unit edge product cluster hypergraph* ***AMS Subject Classification:*** *05C65.*

1 Introduction

The concept of domination in graphs originated with the Queens Problem in 1850. The idea of domination in graphs was introduced by G Claude Berge and Oystein Ore in 1962 [1], [2]. The notation of domination in hypergraphs was introduced by B.D. Acharya in 2007 and further studied by many authors. The domination in edge product hypergraphs was introduced by Kishor F Pawar and Megha M. Jadhav in 2021[3]. Several properties and results regarding an edge product hypergraph and unit edge product hypergraph have been studied. Some upper bound relations are derived. Also, the concept Domination in Edge Product Cluster Hypergraphs has been extended to prove some results.

The main results are proved by using the following theorems:

Theorem 1.1. [5] Let $H = (V_X, E)$be a unit edge product cluster hypergraph with an edge $\in E$. Then $f(e) = 1$ if and only if e must be adjacent to all the edges in H.

Theorem 1.2. [7] Suppose $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ be a unit edge product cluster hypergraph with a unit edge. Then $\gamma(H) \leq |e| - k,\ k$ represents the count of pendant vertices in e.

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Theorem 1.3. [7] Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e containing k pendant vertices. Suppose $e_1, e_2, \dots, e_{m-1}$ be the non-unit edges in H. If $e_i \cap e_j = \varphi$ in $H - e$ for all $1 \le i \ne j \le m - 1$, then $\gamma(H) = |e| - k$.

Theorem 1.4. [7] If $H = (V_X, E)$ is a unit edge product cluster hypergraph with a unit edge e , then e serves as a dominating set for H .

Theorem 1.5. [7] Consider the unit edge product cluster hypergraph $\mathrm{H} = (\mathrm{V_X}, \mathrm{E})$ with a unit edge e. Then any two distinct non-unit edges are non-adjacent if and only if $\gamma(H) = m - 1$.

2 Preliminaries

Definition 2.1 [4] Let X be a nonempty set and let V_X be a subset of $P(X)$ such that $V_X \ne \emptyset$ and $X \subset V_X$. Now E be a multi-set that includes elements from $P(P(X))$ such that (i) $E \ne \emptyset$ (ii) considering each element $e \in E$, there is at least one element $v \in V_X$ such that $v \in e$.

Then $H = (V_X, E)$ is called a *Cluster Hypergraph*, V_X is said to be a vertex set and E is known as a multi-hyper edge set.

Definition 2.2 The vertex degree $d_H(x)$ of x is the number of vertices adjacent to the vertex in the cluster hypergraphs H. The maximum vertex degree of a cluster hypergraph is denoted by $\Delta(H)$ and the minimum vertex degree of cluster hypergraph is denoted by $\delta(H)$.

Definition 2.3 A vertex v in a cluster hypergraph $H = (V_X, E)$, the set $N[v] = \{u \in V_X(H) : u \textit{ is adjacent to } v\} \cup \{v\}$ is called the Closed Neighbourhood of v in H.

Definition 2.4 [5] Let $H = (V_X, E)$ be a cluster hypergraph with vertex set $V_X\,(H)$ and edge set $E(H)$. Let I be the set of positive integers such that $|E| = |I|$. Then any bijection $f: E \to I$ is called an Edge Function of the cluster hypergraph H.

Definition 2.5 [5] The function $F(v) = \prod\{f(e);\ \textit{edge } e \textit{ incident to the vertex } v\}$ on $V_X\,(H)$ is said to be an Edge product Function of the edge function f.

Definition 2.6 [5] The cluster hypergraph $H = (V_X, E)$ is called an Edge Product Cluster Hypergraph if there is an edge function $f: E \to I$ such that the edge function f and the corresponding edge product function F of f on $V_X(H)$ have the following two conditions:

(i) $F(\boldsymbol{v}) \in I$ for every $\boldsymbol{v} \in V_X(H)$

(ii) if $f(e_1) \times f(e_2) \times \ldots \times f(e_i) \in I$ for some edges $e_1, e_2, \ldots e_i \in E(H)$, then the edges $e_1, e_2, \ldots e_i$ are all incident to some vertex $v \in V_X(H)$.

Definition 2.7 [5] Let $H = (V_X, E)$ be an edge product cluster hypergraph. Then H is said to be a Unit Edge Product Cluster Hypergraph if there exists an edge function $f: E \to I$ such that $1 \in I$.

Definition 2.8 [6] Let H be a cluster hypergraph. The minimum cardinality of a maximal strongly independent set in H is called the independent domination number and is denoted by $i(H)$.

3. Main Results

Definition 3.1[7] Let $H = (V_X, E)$ is an edge product cluster hypergraph. A Dominating Set of H is a collection of vertices $S \subseteq V_X(H)$ so that for each vertex $x \in V_X(H) - S$ there is an edge $e \in E(H)$ for which $x \in e$, $e \cap S \neq \emptyset$.

ie, for each vertex $x \in V_X(H) - S$ is adjacent to a vertex in S.

Example 3.1 Consider $H = (V_X, E)$ is an edge product cluster hypergraph where $V_X(H)$=$\{v_1, v_2, v_3, v_4, v_5, \ldots, v_{18}\}$ and $E(H) = \{e_1, e_2, \ldots e_8\}$. The edges of H is defined as $e_1 = \{v_3, v_4, v_8, v_{12}, v_{16}\}$, $e_2 = \{v_1, v_3\}$, $e_3 = \{v_2, v_3\}$, $e_4 = \{v_4, v_5, v_6, v_7, \ldots, v_{11}\}$, $e_5 = \{v_8, v_9, v_{10}, v_{11}\}$, $e_6 = \{v_{12}, v_{13}, v_{14}, v_{15}\}$, $e_7 = \{v_{16}, v_{17}, v_{18}\}$, $e_8 = \{v_{19}, v_{20}\}$. Now the edge function $f: E \to I$ is determined by $f(e_1) = 1$, $f(e_2) = 7$, $f(e_3) = 11$,$f(e_4) = 2$,$f(e_5) = 3$, $f(e_6) = 6$, $f(e_7) = 77$,$f(e_8) = 3$. The edge product function F of f is defined by $F(v_1) = 7, F(v_2) = 11, F(v_3) = 77, F(v_4) = 2, F(v_5) = F(v_6) = F(v_7) = 2, F(v_8) = F(v_9) = F(v_{10}) = F(v_{11}) = F(v_{12}) = F(v_{13}) = F(v_{14}) = F(v_{15}) = 6$, $F(v_{16}) = F(v_{17}) = F(v_{18}) = 77, F(v_{19}) = F(v_{20})$=3.

Here v_3 dominates $\{v_1, v_2\}$, v_4 dominates $\{v_5, v_6, v_7, .., v_{11}\}$, v_{12} dominates $\{v_{13}, v_{14}, v_{15}\}$, v_{16} dominates $\{v_{17}, v_{18}, v_{15}\}$and v_{19} dominates v_{20}.

Therefore, $S = \{v_3, v_4, v_{12}, v_{16}, v_{19}\}$ with cardinality 5 dominates H. Hence, $\gamma(H) = 5$.

Theorem 3.2 Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph with a unit edge e containing k pendant vertices. If the set of edges $e_1, e_2, \ldots, e_{m-1}$ serve as the non-unit edges in H. Then $\gamma(H) = |e| - k$ iff $e_i \not\subseteq e_j$ for all $1 \leq i \neq j \leq m-1$ in $H - e$.

Proof Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph with a unit edge e containing k pendant vertices. Let $e_1, e_2, \ldots, e_{m-1}$ serves as the non- unit edges in H.

First assume that $e_i \not\subseteq e_j$ for all $1 \leq i \neq j \leq m-1$ in $H - e$. Let $e_i' = e_i - e$ and $e_j' = e_j - e$. Then clearly, $e_i' \not\subseteq e_j$ for all $1 \leq i \neq j \leq m-1$ in H.

If $e_i \cap e_j = \emptyset$ in $H - e$, then $\gamma(H) = |e| - k$ (3).

Suppose $e_i \cap e_j = \emptyset$. If $e_i' = \emptyset$, then $e_i \cap e_j$ contains only the vertices in e. The vertices in e dominates the vertices in e_i and e_j. In e the vertices $v_1, v_2, \ldots, v_{m-1}$ must dominate all the other vertices contained in the edges $e_1, e_2, \ldots, e_{m-1}$. Clearly, the the non-pendant vertices in $\overline{e}$ is dominated by pendant vertices in e. So, there are minimum $|e| - k$ number of vertices required to dominate the unit edge product cluster hypergraphs H. So $\gamma(H) \geq |e| - k$. Thus, by Theorem 1.2, it is concluded that $\gamma(H) = |e| - k$.

Similarly, to prove for $e_j' = \emptyset$. Assume that, $e_i' \neq \emptyset$ and $e_j' \neq \emptyset$. Since $e_i' \nsubseteq e_j'$ for all $1 \leq i \neq j \leq m-1$ all the vertices in e_i' is not dominated by the vertices in e_j' . To dominate the vertices in e_i' choose the vertex v_i . choose the vertex v_j, to dominate the vertices in e_j' for all $i \neq j$. Therefore, there exists at least $|e| - k$ vertices in H in order to dominate all the vertices in H, and so $\gamma(H) \geq |e| - k$. By Theorem 1.2, it can be concluded that $\gamma(H) = |e| - k$.

Conversely, suppose that $\gamma(H) = |e| - k$. Let $u_1, u_2, \ldots, u_k$ be the collection of all pendant vertices in H. Let $S = e - \{u_1, u_2, \ldots, u_k\}$ dominates H. Let e_i and e_j be any two non-unit distinct edges in H. To prove $e_i \nsubseteq e_j$ for all $1 \leq i \neq j \leq m-1$. Suppose there exists p, l with $1 \leq p < l \leq m-1$ such that $e_p \subseteq e_l$. Then, the intersection of e_p and e_l is e_l .

Let $x \in e_l \cap e$. Since H serves as a unit edge product cluster hypergraph for every $e_i \in E$, choosing a vertex $x \in e_l \cap e$ for $1 \leq i \leq m-1$. The vertices inside the cluster vertices are dominated by the cluster vertices. Obviously, these vertices must be in the intersection of e_i and the unit edge e. If $x_1, x_2, \ldots, x_s$ be the distinct vertices incident with the edges $e_1, e_2, \ldots, e_{m-1}$ and different from the pendant vertices $u_1, u_2, \ldots, u_k$ together with x forms a dominating set H with cardinality $s+1$ is equal to $|e| - k - 1 < |e| - k$ which leads to a contradiction. Thus, our assumption is wrong. Hence, $e_i \nsubseteq e_j$ for all non-unit edges with $1 \leq i \neq j \leq m-1$.

Theorem 3.3 Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph. Then any minimum dominating set of H must contain vertices from the unit edge.

Proof. Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph with a unit edge e. Let S dominates H. Since H serves a unit edge product cluster hypergraph, by the Theorem 1.1, there exists at least one edge other than e which is adjacent only to e.

Let m be the edge in H which is adjacent to e. Then, there is a vertex $y \in S$ such that x is adjacent to y, for any $x \in e$. Therefore, y belongs to either the intersection of e and m or the pendant vertex in e. Hence, S must contain a vertex from the unit edge e.

Theorem 3.4 Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph with $\gamma(H) + \Delta(H) = |V_X(H)|$ and consider x as a vertex of maximum degree in H. Then, the unit edge e contains at least $|H - N[x]|$ number of vertices.

Proof. Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph with $\gamma(H) + \Delta(H) = |V(H)|$. Consider x be a vertex of maximum degree in H and e be a unit edge in H. It is to be verified that, e contains at least $|H - N[x]|$ number of vertices. Suppose m be an edge that includes the vertex $y \in H - N[x]$. Since H serves as a unit edge product cluster hypergraph, m cannot be a unit edge in H. By the Theorem 1.4, m must be adjacent to e in H. Thus, there is a vertex $u \in e$ such that u is adjacent to y, which belongs to the intersection of the unit edge and the edge containing the vertex y, for every vertex $y \in H - N[x]$. To show that, $|e| \geq |H - N[x]|$. Suppose on contrary, assume that $|e| < |H - N[x]|$. This shows that there doesn't exist a unique vertex $u \in e$ with $y \in H - N[x]$.

Let $x_1, x_2 \in H - N[x]$ there exists a single vertex $u \in e$ with $S = \{\{x, u\} \cup (V - [N[x] \cup (N(u) \cap (V - N[x])))]\}$ dominates H with cardinality $2 + |V_X(H)| - \Delta(H) - 1 - 2 = |V_X(H)| - \Delta(H) - 1$. It follows that $\gamma(H) + \Delta(H) = |V_X(H)| - 1$, which is a contradiction. Hence, the unit edge e contains at least $|H - N[x]|$ number of vertices.

Theorem 3.5. Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph with $i(H) + \Delta(H) = |V_X(H)|$ and consider x as a vertex of maximum degree in H. Then H contains at least $|V_X(H) - N[x]|$ number of non-unit edges in H.

Proof. Let $H = (V_X, E)$ be a unit edge product cluster hypergraph with $i(H) + \Delta(H) = |V_X(H)|$ and let x be a vertex of maximum degree in H. Let e be a unit edge in H. Suppose H contains at most $|V_X(H) - N[x]| - 1$ number of non-unit edges in H. Since H is a unit edge product cluster hypergraph, any edge containing the vertex $y \in V_X(H) - N[x]$ is adjacent to the edge e. Clearly, there exists two vertices u and v in $V_X(H) - N[x]$ with u and v are adjacent. Therefore, the magnitude of any strongly independent dominating set of $V_X(H) - N[x]$ is at most $|V_X(H) - N[x] - 1|$.

Thus, $i(H) \leq |V_X(H) - N[x]| - 1 + |x| = |V_X(H)| - \Delta(H) - 1$. That is, $i(H) + \Delta(H) \leq |V_X(H)| - 1$ implies that $i(H) + \Delta(H) < |V_X(H)|$, which is a contradiction. Hence our assumption is wrong. Therefore, H contains minimum $|V_X(H) - N[x]|$ number of non-unit edges in H.

Theorem 3.6 Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph with a unit edge e containing at least one pendant vertex. If $\gamma(H) = m - 1$, then each dominating set contains at least one vertex form e.

Proof. Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph with a unit edge e containing at least one pendant vertex. Assume $\gamma(H) = m - 1$. Let S be any dominating set in H. To prove S contains at least one vertex e. Suppose S contains no elements from e. Since $\gamma(H) = m - 1$, S contains at most $m - 1$ number of vertices. Since H contains m edges and each non-unit edges are adjacent to e and S contains no vertex from e, so that S must contain vertices from the non-unit edges. To dominate the pendant vertex in e and the non-unit vertices in H, there must be any two non-unit edges are adjacent as it is $|S| \leq m - 1$. By a Theorem 1.5, $\gamma(H) \neq m - 1$, this results in a contradiction. So, our assumption is incorrect. Therefore, S contains at least one vertex from the unit edge e.

Theorem 3.7 Let $H = (V_X, E)$ represents a unit edge product cluster hypergraph of size $m \geq 3$. If e is a unit edge of H then,

(i) $2 \leq \gamma(H) + \gamma(\overline{H}) \leq |e| + 1$.

(ii) $1 \leq \gamma(H).\gamma(\overline{H}) \leq |e|$.

Proof. For any unit edge product cluster hypergraph H is a cluster hypergraph. Then $\gamma(H) \geq 1$. Also H is connected, the complement $\overline{H}$ is also a cluster hypergraph and $\gamma(\overline{H}) \geq 1$. This suggests that $\gamma(H) + \gamma(\overline{H}) \geq 2$ and $\gamma(H).\gamma(\overline{H}) \geq 1$. Thus, it proves the lower bound.

In order to prove for the upper bound, consider that H is a cluster hypergraph, then the unit edge forms a dominating set in H. Therefore, $\gamma(H) \leq |e|$. Every maximal edge in H is adjacent only to e. Let m be the edge in H, which is adjacent only to e. Let f be any edge in H. So, the edges m and f are two independent edges in H. Consider the vertex $x \in V_X(H)/m \cup f$, then for any other vertex $y \in V_X(H)$ such that the vertex x is adjacent to y in $\overline{H}$. Thus $\gamma(\overline{H}) \leq 1$. Hence, it follows that $\gamma(H) + \gamma(\overline{H}) \leq |e| + 1$ and $\gamma(H).\gamma(\overline{H}) \leq |e|$.

4. Conclusion

In this article, the concept Domination in Edge Product Cluster Hypergraph and Unit Edge Product Cluster Hypergraph has been applied and also, the same concept is extended to prove some theorems related to the Domination Number in Edge Product Cluster Hypergraph, Unit Edge Product Cluster Hypergraph.

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